THEOREM OF THE CUBE

$H^0(X(s), \mathcal{O}_{X(s)}) = k(s)$ (A)

Theorem: Let $p: X \to S$ be a [proper, flat and geometrically integral] morphism. Let $\mathcal{L} \in \mathrm{Pic}(X)$, there is a closed subscheme, $Z \hookrightarrow S$ st $\forall T \to S$, $T \to Z \to S$ factors $\iff \mathcal{L}|_{X_T} \cong p_T^* M$, $M \in \mathrm{Pic}(T)$.

1. Picard Functor:

Definition: Let $X \xrightarrow{p} S$ be an (A) morphism define $\mathrm{Pic}_{X/S}(S) = \mathrm{Pic}(X)/p^*\mathrm{Pic}(S)$ extend this to a functor
$$\mathrm{Pic}_{X/S}: \mathrm{Sch}_{/S}^{op} \longrightarrow \mathrm{Ab}$$
$$T \longmapsto \mathrm{Pic}_{X_T/T}(T).$$

Terminology: Let $X \to S$ be an (A)-morphism.

- A l.b $\mathcal{L} \in \mathrm{Pic}(X)$ is S-trivial if $\mathcal{L} \equiv 0$ in $\mathrm{Pic}_{X/S}$ ($\mathcal{L} = p^*M$ $M \in \mathrm{Pic}(S)$).
- $\mathcal{L}_1 \equiv_S \mathcal{L}_2$ if $\mathcal{L}_1 = \mathcal{L}_2$ in $\mathrm{Pic}_{X/S}$.

Remark: A l.b $\mathcal{L} \in \mathrm{Pic}(X)$ is S-trivial $\iff p_*\mathcal{L} \in \mathrm{Pic}(S)$ and $p^*p_*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$

Proof: For (A)-morphisms, we have $p_*\mathcal{O}_X = \mathcal{O}_S$. If $\mathcal{L}$ is S-trivial, $\mathcal{L} \equiv p^*M$ $M \in \mathrm{Pic}\,S$. Goal: $p_*\mathcal{L} = M$. $p^*\mathcal{L} = p^*p_*\mathcal{L} = M$, as locally on S, $M \cong \mathcal{O}_S$ we have $p_*p^*\mathcal{O}_S = p_*\mathcal{O}_X = \mathcal{O}_S \Rightarrow p_*\mathcal{L} = M$.

Theorem (Grothendieck): (to be proved later)
$\mathrm{Pic}_{X/S}$ is representable.

Remark: Seesaw theorem just says that $\mathrm{Pic}_{X/S}$ is separated.
$\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}_{X/S}(S)$, $Z(T) = \{T \to S \mid \mathcal{L}_1|_{X_T} \equiv \mathcal{L}_2|_{X_T} \bmod T\}$ is a closed subscheme.

$$\begin{tikzcd} Z_T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "(\mathcal{L}_1, \mathcal{L}_2)"] \\ \mathrm{Pic}_{X/S} \arrow[r, "\Delta"] & \mathrm{Pic}_{X/S} \times \mathrm{Pic}_{X/S} \end{tikzcd}$$

Remark: Z in Seesaw is
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\mathcal{L}"] \\ S \arrow[r, "1"'] & \mathrm{Pic}_{X/S} \end{tikzcd}$$

Theorem of the Cube:
Let $X, Y \to S$ be (A)-morphisms. S is connected. Let $x \in X(S)$, $y \in Y(S)$. Let $\mathcal{L} \in \mathrm{Pic}(X \times_S Y)$ s.t. $\exists s \in S$ with $\underbrace{\mathcal{L}|_{X_s \times_{k(s)} Y_s} \text{ is trivial.}}_{(1)}$

$$\Rightarrow \mathcal{L} \cong S\, \iota^*_{X \times \{y\}} \mathcal{L}|_{X \times \{y\}} \otimes \iota^*_{\{x\} \times Y} \mathcal{L}|_{\{x\} \times Y} \bmod S \quad \text{where } \iota_{X \times \{y\}}: X \times_S Y \longrightarrow X \times_S \{y\}.$$

Proof: By Seesaw let $Z \hookrightarrow S$ where $M = \mathcal{L}^{-1} \otimes \iota^*_{X \times \{y\}} \mathcal{L}|_{X \times \{y\}} \otimes \iota^*_{\{x\} \times Y} \mathcal{L}|_{\{x\} \times Y}$

- First $Z \neq \emptyset$ by (1).

As S is connected, Z open will conclude. To show that Z is open, $\forall z \in Z$ $\mathrm{Spec}(\mathcal{O}_{S,z}) \longrightarrow Z$, if $Z = V(I)$ it means that $\mathcal{O}_{S,z}.I = 0$, it suffices to show $\mathcal{O}_{S,z}/m_z^n . I = 0$ ($\Rightarrow I \subseteq \bigcap_n m_z^n = 0$).
To simplify the argument, let $a_n \subseteq \mathcal{O}_{S,z}$ sequence of ideals, $\bigcap a_n = 0$ and $a_n/a_{n+1} \cong k(s)$.

Goal: $\mathcal{O}_{S,z}/a_n . I = 0$ $\forall n$. Using universal property of $V(I) = Z$ same as showing that $M|_{V_n}$ is trivial. $S_n = \mathrm{Spec}(\mathcal{O}_{S,z}/a_n)$, $V_n = S_n \times_S (X \times_S Y)$.

Induction on n: For $n=1$ by assumption (1) $\mathcal{O}_{V_1} \xrightarrow[\sim]{\theta_1} M_{V_1}$, if I can lift θ_1 to a $\theta_n: \mathcal{O}_{V_n} \longrightarrow M|_{V_n}$ by Nakayama θ_n would be an iso.

$0 \longrightarrow a_n \cdot M_{n+1} \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0$ les in cohomology: $H^0(V_{n+1}, M_{n+1}) \longrightarrow H^0(V_n, M_n) \ni \theta_n \xrightarrow{\delta} H^1(V_1, \mathcal{O}_{V_1})$

$a_n \cdot M_{n+1} \cong \mathcal{O}_{V_1}$ (by simplifying trick)

θ_n lifts $\Leftrightarrow \delta(\theta_n) = 0$

$$\begin{tikzcd}
H^0(V_n, M_n) \arrow[rr, "\delta"] \arrow[d, "\cong"'] \arrow[drr, phantom, "\circlearrowright"] & & H^1(V_1, \mathcal{O}_{V_1}) \arrow[d, "\cong", "\text{Künneth}"'] \\
H^0((X\times\{y\})|_{S_n}, M_n|_-) \oplus H^0(\{x\}\times Y|_{S_n}, M_n|_-) \arrow[rr, "\delta_1"] & & H^1(X\times\{y\}|_{S_1}, -) \oplus H^1(\{x\}\times Y|_{S_1}, -)
\end{tikzcd}$$

On both $\{x\}\times Y$ and $X\times\{y\}$ M is trivial $\Rightarrow$ the lifting problem for both these spaces and (M_n) has a solution $\Rightarrow \delta_1(\theta_n|_{\{y\}\times X}) = \delta_2(\theta_n|_{Y\times\{x\}}) = 0$ $\square$

Remark: Theorem of the cube $\Leftrightarrow \mathrm{Pic}^\circ_{X\times Y/k} \cong \mathrm{Pic}^\circ_{X/k} \times \mathrm{Pic}^\circ_{Y/k}$

Corollary: Let $k = \bar{k}$, let $X, Y / k$ be proper schemes and Z/k any connected scheme. For any $\mathcal{L} \in \mathrm{Pic}(X\times Y\times Z)$, $\mathcal{L} \cong p^*_{X\times Y}\mathcal{L}_1 \otimes p^*_{X\times Z}\mathcal{L}_2 \otimes p^*_{Y\times Z}\mathcal{L}_3$ where $p_{X\times Y} : X\times Y\times Z \to X\times Y$ is the projection.

Proof: $X\times Y\times Z \longrightarrow Z$ and apply cube to $\mathcal{L}^{-1} \otimes p^*_{X\times Y}\mathcal{L}|_{X\times Y\times\{z\}}$.

END OF LECTURE 10